ISRAEL JOURNAL OF MATHEMATICS 102 (1997), 29–60

SEMISMALL PERTURBATIONS IN THE MARTIN THEORY FOR ELLIPTIC EQUATIONS

BY

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ABSTRACT

We investigate stability of Martin boundaries for positive solutions of elliptic partial differential equations. We define a perturbation which is G_L^D-semismall at infinity, show that Martin boundaries are stable under this perturbation, and give sufficient conditions for it.

§1. Introduction

This paper is concerned with a perturbation theory in the Martin theory for the structure of all positive solutions of a second order elliptic partial differential equation. The perturbation theory is not only important in itself, but also crucial in studying the structure of positive solutions by exploiting the separation of variables method (cf. [AM], [M1,2], and references therein).

The aim of this paper is to make it clear what is a "small" perturbation in the Martin theory for positive solutions of an elliptic partial differential equation $Lu = 0$ in a domain $D \subset \mathbb{R}^n$.

We shall introduce the notion of G_L^D-semismallness at infinity, and show that Martin boundaries are stable under perturbations which are G_L^D-semismall at infinity (see Definition 1.1 and Theorem 1.4 below). Here the term "at infinity"

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* This work was carried out partially while the author was visiting the Department of Mathematics, Technion — Israel Institute of Technology. He wishes to thank the Israel Association for the Promotion of International Scientific Relations, the Japan Society of Promotion of Science, and the Technion.
Received November 28, 1995 and in revised form August 1, 1996

means "near an infinity point in the one point compactification of D". After giving sufficient conditions for a perturbation to be G_L^D-semismall at infinity, which are easy to check, we shall give concrete examples by applying them (see Theorems 4.2, 5.1, 5.4.1, 5.9, and 5.11 in Sections 4 and 5). Notions and results related to G_L^D-semismallness will also be discussed in this paper.

As for the Martin theory for elliptic equations and perturbation theories for it, see [Ag], [AM], [BHH], [CC], [CFZ], [H], [HZ], [L], [LP], [Mae], [Mar], [M1,2,3,4], [N], [P1,2,3], [Pi], [T], [Z1,2,3], and a brief explanation before Theorem 1.4 to be stated below.

Let $n \geq 2$ and $p > n/2$. Let L be an elliptic operator on a domain D in $\mathbb{R}^n$ of the form

$$L = -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j) - \sum_{j=1}^{n} b_j(x)\partial_j + V(x), \tag{1.1}$$

where $\partial_j = \partial/\partial x_j$, b_j $(j = 1, \ldots, n)$ and V are real-valued functions in $L_{2p,loc}(D)$ and $L_{p,loc}(D)$, respectively, and $(a_{ij}(x))_{i,j=1}^n$ is a positive definite symmetric matrix-valued measurable function on D such that for any compact set K in D there exists a positive constant Λ with

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x \in K, \quad \xi \in \mathbb{R}^n.$$

Throughout the present paper we assume that (L, D) is subcritical, i.e., there exists the (minimal positive) Green function G_L^D for (L, D). Let $\{D_j\}_{j=1}^\infty$ be an increasing sequence of smooth bounded domains in D such that $D_j \Subset D_{j+1}, j = 1, 2, \ldots,$ and $\bigcup_{j=1}^\infty D_j = D$, where $D_j \Subset D_{j+1}$ means that the closure $\overline{D_j}$ of D_j is a compact subset of D_{j+1}. Then

$$G_L^D = \lim_{j\to\infty} G_j, \tag{1.2}$$

where G_j is the Green function for L in D_j with zero Dirichlet boundary condition (cf. [M2] and [S]).

Let ${}^tL$ be the formal adjoint operator of L:

$${}^tL = -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j) + \sum_{j=1}^{n} \partial_j(b_j) + V = 0.$$

Recall that the (minimal positive) Green function $G^D_{{}^tL}$ for $({}^tL, D)$ is given by

$$G^D_{{}^tL}(x,y) = G^D_L(y,x), \quad (x,y) \in D^2.$$

For an open subset Ω of D, $H^1_{loc}(\Omega)$ (or $H^1(\Omega)$) denotes the set of all functions f in $L_{2,loc}(\Omega)$ (or $L_2(\Omega)$) whose first order distributional derivatives $\partial_j f$ belong to $L_{2,loc}(\Omega)$ (or $L_2(\Omega)$); and $H^1_0(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in the Sobolev space $H^1(\Omega)$. A solution u of the equation $Lu = 0$ in Ω means a function $u \in H^1_{loc}(\Omega)$ satisfying

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi - \sum_{j=1}^n b_j \partial_j u \varphi + V u \varphi \right) dx = 0$$

for any $\varphi \in C_0^\infty(\Omega)$; and a solution u of the equation ${}^tLu = 0$ in Ω is defined similarly. It is known (cf. [S]) that any solution u of $Lu = 0$ (or ${}^tLu = 0$) in Ω is continuous. A supersolution u of the equation $Lu = 0$ in Ω means a real-valued function $u \in H^1_{loc}(\Omega)$ such that $Lu \geq 0$ in Ω in the weak sense. Following S. Agmon, L is said to be δ-positive in $\Omega \Subset D$ when any supersolution u of $Lu = 0$ in Ω such that

$$u \geq 0 \text{ on } \partial\Omega \quad (\text{i.e., } u_-(x) = \max(-u(x), 0) \in H^1_0(\Omega))$$

is nonnegative a.e. in Ω. It follows from the subcriticality of (L, D) that L is δ-positive in any open set $\Omega \Subset D$ (cf. [M2, Theorem 1.5]). This positivity property of supersolutions is equivalent to the weak maximum principle if $V = 0$; and, by abusing the terminology, we shall call it the maximum principle. This maximum principle plays a basic role in studying positive solutions; for example, the proof of (1.2) is based upon the monotonicity $G_j \leq G_{j+1}$, which follows from this maximum principle.

Let x_0 be a point in D_1 (it is fixed and called a reference point), and W be a real-valued function in $L_{p,loc}(D)$. Put $G = G^D_L$.

Definition 1.1: We say that W is G^D_L-semismall at infinity when

$$\lim_{j\to\infty} \sup_{z\in D_j^c} \frac{1}{G(x_0,z)} \int_{D_j^c} G(x_0,y)|W(y)|G(y,z)dy = 0, \tag{1.3}$$

where $D_j^c = D \setminus D_j$. We say that W is $G^D_{{}^tL}$-semismall at infinity when

$$\lim_{j\to\infty} \sup_{z\in D_j^c} \frac{1}{G(z,x_0)} \int_{D_j^c} G(z,y)|W(y)|G(y,x_0)dy = 0. \tag{1.3'}$$

We rename the small perturbation introduced by Pinchover [P2] as follows.

Definition 1.2: We say that W is G_L^D-small at infinity when

$$\lim_{j\to\infty} \sup_{x,z\in D_j^c} \frac{1}{G(x,z)} \int_{D_j^c} G(x,y)|W(y)|G(y,z)dy = 0. \tag{1.4}$$

The following proposition says that G_L^D-semismallness is, indeed, a semi-version of G_L^D-smallness.

PROPOSITION 1.3: *If W is G_L^D-small at infinity, then it is both G_L^D-semismall at infinity and $G_{^tL}^D$-semismall at infinity.*

This proposition and theorems below will be proved in Section 2.

We denote by $D_L^*, \partial_M D_L, \partial_m D_L, K_L^D$, and $H_+(L,D)$, the Martin compactification, Martin boundary, minimal Martin boundary, Martin kernel for (L,D), and the cone of positive solutions of the equation $Lu = 0$ in D, respectively. For their definitions and basic properties, see [BHH], [CC], [H], [Mae], [Mar], [M1,2], and [Pi]. Here, we only recall that $D_L^* = D \cup \partial_M D_L$ is a compact metric space including D as an open dense subset, for any $u \in H_+(L,D)$ there exists a unique finite Borel measure on $\partial_M D_L$ such that $\mu(\partial_M D_L \setminus \partial_m D_L) = 0$ and

$$u(x) = \int_{\partial_M D_L} K_L^D(x,\xi)d\mu(\xi),$$

and a point ξ in the Martin boundary $\partial_M D_L$ is an equivalence class of fundamental sequences $\{y_j\}_{j=1}^\infty$ in D: (i) $\{y_j\}_{j=1}^\infty$ has no accumulation points in D; (ii) $G(x,y_j)/G(x_0,y_j)$ converges uniformly on any compact subset of D to a positive solution (which is the Martin kernel $K_L^D(x,\xi)$); and (iii) two fundamental sequences are said to be equivalent if their limits are identically equal. We write $K = K_L^D$.

THEOREM 1.4: *Suppose that $(L+W,D)$ is subcritical. Put $G_W = G_{L+W}^D$ and $K_W = K_{L+W}^D$. Assume that W is G_L^D-semismall at infinity. Then there exists a homeomorphism Φ from D_L^* onto D_{L+W}^* such that $\Phi|_D =$ identity and $\Phi(\partial_m D_L) = \partial_m D_{L+W}$. Furthermore, a linear operator T defined by*

$$Tu(x) = u(x) - \int_D G_W(x,z)W(z)u(z)dz \tag{1.5}$$

is a continuous order preserving linear bijection from $H_+(L,D)$ onto $H_+(L+W,D)$,

$$K_W(x,\Phi\xi) = \frac{TK(x,\xi)}{TK(x_0,\xi)}, \quad \xi \in \partial_M D_L, \tag{1.6}$$

the Green function G_W satisfies the resolvent equation

$$G_W(x,y) = G(x,y) - \int_D G_W(x,z)W(z)G(z,y)dz, \tag{1.7}$$

and is semi-comparable with G, i.e., for any compact set F in D there exists a positive constant C such that

$$C^{-1}G(x,y) \le G_W(x,y) \le CG(x,y), \quad (x,y) \in F \times D. \tag{1.8}$$

Remark: Nakai ([N]) investigated a similar problem for Schrödinger equations on a Riemann surface R. It is easily seen that the assumption of his main theorem implies that the difference $P-Q$ of two potentials is $G^R_{-\Delta+P}$- semismall at infinity (rigourously speaking, in the sense of a Riemann surface version). Thus his main theorem is, in a sense, a corollary of our Theorem 1.4.

Recall that $(L+W, D)$ is said to be critical if the Green function for $(L+W, D)$ does not exist, but there exists a positive solution u of $(L+W)u = 0$ in D, and that in the critical case a positive solution is unique up to a constant multiple (cf. [M2]). The following theorem is a generalization of [M2, Theorem 1.9], and is essentially known in the special case where W is G^D_L-small at infinity (cf. [P3, Lemma 2.5]).

THEOREM 1.5: *Suppose that $(L+W, D)$ is critical, and let u be a positive solution of $(L+W)u = 0$ in D. Assume that W is $G^D_{{}^tL}$-semismall at infinity. Then there exists a positive constant C such that*

$$C^{-1}G(x,x_0) \le u(x) \le CG(x,x_0), \quad x \in D \setminus D_1. \tag{1.9}$$

Furthermore, u satisfies the integral equation

$$u(x) = -\int_D G(x,y)W(y)u(y)dy. \tag{1.10}$$

Surprisingly, the following result is new.

THEOREM 1.6: *Suppose that $(L+W, D)$ is subcritical, and W is G^D_L-small at infinity. Then all the conclusions of Theorem 1.4 are valid. Furthermore, there exists a positive constant C such that*

$$C^{-1}K(x,\xi) \le K_W(x,\Phi\xi) \le CK(x,\xi), \quad x \in D, \quad \xi \in \partial_M D_L, \tag{1.11}$$

$$C^{-1}G(x,y) \le G_W(x,y) \le CG(x,y), \quad (x,y) \in D \times D. \tag{1.12}$$

For a special case when the support of W is compact, this theorem can be generalized as follows. For relevant results, see [L], [T], [M1], and [P1].

THEOREM 1.7: *Suppose that an elliptic operator L' on D is of the same form as L, and $L' = L$ on $D \setminus F$ for a compact subset F of D. Assume that (L', D) is also subcritical, and denote by G' and K' the Green function and the Martin kernel for (L', D), respectively. Then there exists a continuous order preserving linear bijection T from $H_+(L, D)$ onto $H_+(L', D)$, a homeomorphism Φ from D_L^* onto $D_{L'}^*$ satisfying $\Phi|_D =$ identity and $\Phi(\partial_m D_L) = \partial_m D_{L'}$, and a positive constant C such that*

$$C^{-1}G(x,y) \le G'(x,y) \le CG(x,y), \quad (x,y) \in D \times D, \tag{1.13}$$

$$C^{-1}u(x) \le Tu(x) \le Cu(x), \quad x \in D, \quad u \in H_+(L,D), \tag{1.14}$$

$$C^{-1}K(x,\xi) \le K'(x,\Phi\xi) \le CK(x,\xi), \quad x \in D, \quad \xi \in \partial_M D_L, \tag{1.15}$$

$$K'(x,\Phi\xi) = \frac{TK(x,\xi)}{TK(x_0,\xi)}, \quad \xi \in \partial_M D_L. \tag{1.16}$$

The remainder of this paper is organized as follows. In Section 2, we prove the theorems and proposition of this section. In Section 3, notions related to G_L^D-semismallness are introduced and their relations are established. In Section 4, two sufficient conditions for a perturbation to be G_L^D-semismall at infinity are given. The first one comes from the basic estimate which grew in the study of life time estimates, while the second one comes from the 3G theorem which grew in studying conditional gauge theorems (cf. [Ai], [AM], [B1,2,3], [BC], [BD1,2], [BHH], [BØ], [CFZ], [CM], [F], [Z1,2,3]). Applying these sufficient conditions, we give several concrete examples concerning G_L^D-semismallness in Section 5. Among others, Example 5.10 reads as follows.

Example: Suppose that

$$L = -\sum_{i,j=1}^{n} \partial_i a_{ij}(x)\partial_j + V(x),$$

where the coefficients a_{ij} and V are periodic functions on $\mathbb{R}^n$, and $\lambda \le V \le \mu$ for some positive constants λ and μ. Assume that W satisfies, for a sufficienntly large natural number N,

$$\sum_{k=N}^{\infty} \sup_{\frac{k}{N}<|w|<Nk} \int_{|y-w|<1} E(y,w)|W(y)|dy < \infty,$$

where $E(y,w) = |y-w|^{2-n}$ for $n \ge 3$, and $E(y,w) = \log(2/|y-w|)$ for $n = 2$. Then W is $G_L^{\mathbb{R}^n}$-semismall at infinity. Furthermore, if $(L+W, \mathbb{R}^n)$ is subcritical, then $\partial_m D_{L+W}$ is homeomorphic to the boundary of a non-empty strictly convex bounded open subset of $\mathbb{R}^n$.

§2. Proof of Theorems 1.4–1.7 and Proposition 1.3

Proof of Proposition 1.3: Proposition 1.3 follows directly from the following lemma. ∎

LEMMA 2.1: *Suppose that there exists a positive constant ϵ such that*

$$\int_{D_j^c} G(x,y)|W(y)|G(y,z)dy \le \epsilon G(x,z) \tag{2.1}$$

for any $x, z \in D_j^c$. *Then* (2.1) *holds for any* $x, z \in D$.

Proof: We claim that (2.1) holds for any (x, z) in $D_j \times (D \setminus \overline{D_j})$. Fix z in $D \setminus \overline{D_j}$. In view of the monotone convergence theorem, it suffices to show that

$$v(x) \equiv \int_{D_j^c} G(x,y)f(y)dy \le \epsilon G(x,z), \quad x \in D_j,$$

where f is a nonnegative measurable function on D such that it has compact support in $D \setminus \overline{D_j}$, and is bounded from above by the minimum of a positive constant and $|W(y)|G(y,z)$. Put $u(x) = \epsilon G(x,z) - v(x)$. Then, for some open set $\Omega \supset \overline{D_j}$, $u \in H^1(\Omega) \cap C^0(\Omega)$, $u \ge 0$ on $\Omega \setminus D_j$, and $Lu = 0$ in Ω. Thus u is a solution of the equation $Lu = 0$ in D_j satisfying $u \ge 0$ on ∂D_j in the sense that $u_- \in H_0^1(D_j)$. Therefore the maximum principle (or δ-positivity of L in D_j) shows that $u(x) \ge 0$ for any $x \in D_j$. This proves the claim. Now, fix x in D_j. Since the left and right hand sides of (2.1) are both continuous on D_j^c with respect to z, we then get (2.1) for any $(x, z) \in D_j \times D_j^c$. Similarly, (2.1) holds for any $(x, z) \in D_j^c \times D_j$. Finally, fix z in D_j. Then the left hand side of (2.1) is a solution of $Lu = 0$ in D_j, and $L(\epsilon G(\cdot, z)) \ge 0$ in D_j. Thus the maximum principle shows that (2.1) holds for any x in D_j. This completes the proof. ∎

Proof of Theorem 1.7: We first show (1.13). Suppose that $F \subset D_m$. Since (L, D) is subcritical, we can choose a positive solution h of the equation $Lh = 0$ in D. Put

$$L_h = -h^{-2} \sum_{i,j=1}^{n} \partial_i(h^2 a_{ij} \partial_j) - \sum_{j=1}^{n} b_j \partial_j, \tag{2.2}$$

and $G^h = G^D_{L_h}$. Then we see that $G(x,y) = h(x)G^h(x,y)h(y)^{-1}$. We obtain (cf. [GW], [LSW], [S]) that

$$C^{-1}E(x,y) \le G^{D_{m+2}}_{L_h}(x,y) \le CE(x,y), \quad x,y \in \overline{D_{m+1}}, \tag{2.3}$$

where C is a positive constant,

$$E(x,y) = |x-y|^{2-n} \text{ for } n \ge 3, \quad E(x,y) = \log(R/|x-y|) \text{ for } n = 2$$

with $R > \sup\{|x-y|; x,y \in \overline{D_{m+2}}\}$. Let $H_y(x)$ be the solution of the equation $L_h u = 0$ in D_{m+2} with $u(\cdot) = G^h(\cdot,y)$ on ∂D_{m+2}. Since

$$0 < \inf\{G^h(x,y); x \in \partial D_{m+2}, y \in \overline{D_{m+1}}\},$$
$$\sup\{G^h(x,y); x \in \partial D_{m+2}, y \in \overline{D_{m+1}}\} < \infty,$$

the maximum principle implies that $C^{-1} \le H_y(x) \le C$ for any $x \in \overline{D_{m+2}}$ and $y \in \overline{D_{m+1}}$. Since $G^{D_{m+2}}_{L_h}(x,y) = G^h(x,y) - H_y(x)$, this together with (2.3) shows that

$$C^{-1}E(x,y) \le G^h(x,y) \le CE(x,y), \quad x \in \overline{D_{m+2}}, \ y \in \overline{D_{m+1}}. \tag{2.4}$$

Since h is positive continuous on D, it follows from (2.4) that

$$C^{-1}E(x,y) \le G(x,y) \le CE(x,y), \quad x \in \overline{D_{m+2}}, \ y \in \overline{D_{m+1}}. \tag{2.5}$$

Similarly, (2.5) holds with G replaced by G'. Thus

$$C^{-1}G(x,y) \le G'(x,y) \le CG(x,y), \tag{2.6}$$

for any $x,y \in \overline{D_{m+1}}$. Recall that G and G' are minimal positive Green functions for (L,D) and (L',D), respectively, and that $L = L'$ on $D \setminus D_m$. Therefore,

$$[\inf\{G'(z,w); z \in \partial D_{m+1}, w \in \overline{D_m}\}]G(x,y)$$
$$\le [\sup\{G(z,w); z \in \partial D_{m+1}, w \in \overline{D_m}\}]G'(x,y)$$

for any $x \in D \setminus \overline{D_{m+1}}$ and $y \in \overline{D_m}$. This implies that (2.6) holds also for any $(x,y) \in (D \setminus \overline{D_{m+1}}) \times \overline{D_m}$; similarly, (2.6) holds for any $(x,y) \in \overline{D_m} \times (D \setminus \overline{D_{m+1}})$. Since $L(CG(\cdot,y) - G'(\cdot,y)) \ge 0$ in $D \setminus \overline{D_m}$ for any $y \in D \setminus \overline{D_{m+1}}$, we thus get (2.6) for any $(x,y) \in (D \setminus \overline{D_m}) \times (D \setminus \overline{D_{m+1}})$. Similarly, (2.6) holds for any $(x, y) \in (D \setminus \overline{D_{m+1}}) \times (D \setminus \overline{D_m})$. Combining these estimates, we finally get (1.13).

We next show (1.14)–(1.16) briefly along the line given in the proof of Theorem 7.2 in [M2]. Choose $\phi \in C_0^\infty(D_{m+2})$ such that $\phi = 1$ in D_{m+1}. Put $\Omega = D \setminus \overline{D_m}$ and $g = G_L^\Omega$. Define an operator B on $H_+(L, D)$ by

$$Bu(x) = \phi u(x) - \int_\Omega g(x, y) L(\phi u)(y) dy. \tag{2.7}$$

We see that $L(Bu) = 0$ in Ω, $Bu = u$ on $\partial\Omega$, $Bu \geq 0$ in Ω, and $Bu(x) \leq Cg(x, y_0)$ for any $x \in D \setminus \overline{D_{m+3}}$ with y_0 being a point on ∂D_{m+1}. Put

$$H_{+,\infty} = \{u \in H_+(L, \Omega) \cap C(\overline{\Omega} \cap D); u = 0 \text{ on } \partial\Omega \cap D\}.$$

Then the operator S defined by

$$Su = u - Bu \tag{2.8}$$

is a continuous order preserving linear bijection from $H_+(L, D)$ onto $H_{+,\infty}$ (cf. [M2]). Similarly, define an operator S' from $H_+(L', D)$ onto $H_{+,\infty}$. Set $T = (S')^{-1} \circ S$. Then T is a continuous order preserving linear bijection from $H_+(L, D)$ onto $H_+(L', D)$. Since the Harnack inequality yields the inequality

$$C^{-1} u(x) \leq Su(x), \quad x \in D \setminus \overline{D_{m+1}}, \ u \in H_+(L, D),$$

and the corresponding one for S', we get (1.14). It remains to show (1.15) and (1.16). Suppose that $\{y_j\}_{j=1}^\infty$ is a fundamental sequence representing a Martin boundary point $\xi \in \partial_M D_L$. Then we see that it is also a fundamental sequence for (L', D) and the corresponding equivalence class $\xi' \in \partial_M D_{L'}$ is independent of a choice of representatives of ξ. Thus we can define Φ by $\Phi(y) = y$ for $y \in D$, and $\Phi\xi = \xi'$ for $\xi \in \partial_M D_L$. We then get (1.16), which together with (1.14) shows (1.15) and the desired properties of Φ. ∎

Proof of Theorem 1.4: We write $W_j = \chi_j W$, where χ_j is the characteristic function of the set D_j^c. In view of the Harnack inequality, we can choose, for any positive integer m, J so large that the inequality

$$\int_D G(x, y) |W_j(y)| G(y, z) dy \leq \frac{1}{4} G(x, z) \tag{2.9}$$

holds for any $j \geq J$, $x \in D_m$ and $z \in D_j^c$. As in the proof of Lemma 2.1, we see that (2.9) holds also for any $x \in D_m$ and $z \in D$. Now, let us show that the Green function exists for $(L + W_j, D)$, $j \geq J$. Define $H_k(x,z), k = 0, 1, \ldots,$ on $D_m \times D$ by

$$H_0(x,z) = G(x,z), \quad H_k(x,z) = \int_D H_{k-1}(x,y)W_j(y)G(y,z)dy, \ \ k = 1, 2, \ldots.$$

Then, by induction,

$$|H_k(x,z)| \leq 4^{-k}G(x,z), \quad (x,z) \in D_m \times D. \tag{2.10}$$

Put

$$H(x,z) = \sum_{k=0}^{\infty}(-1)^k H_k(x,z). \tag{2.11}$$

Clearly,

$$\tfrac{2}{3}G(x,z) \leq H(x,z) \leq \tfrac{4}{3}G(x,z), \quad (x,z) \in D_m \times D. \tag{2.12}$$

We claim that $H(x,\cdot)$ is the Green function for $({}^tL + W_j, D)$ with pole at x, where ${}^tL$ is the formal adjoint operator of L. For any $\phi \in C_0^\infty(D_m)$, put

$$v_k(z) = \int_D H_k(x,z)\phi(x)dx, \quad v(z) = \int_D H(x,z)\phi(x)dx.$$

We have ${}^tLv_0 = \phi$, and ${}^tLv_k = W_j v_{k-1}$. The local a priori estimates (cf. [S]) and (2.10) yield

$$\|v_k\|_{L_\infty(D_\ell)} + \|v_k\|_{H^1(D_\ell)} \leq C_\ell 4^{-k}, \quad \ell = 1, 2, \ldots,$$

where C_ℓ is a constant depending only on ℓ. Thus, $v \in H^1_{loc}(D)$, and $({}^tL + W_j)v = \phi$. This implies the existence of the Green function $G_{W_j}(x,\cdot)$ for $({}^tL + W_j, D)$ with pole at x. Furthermore, making use of of the approximation by the Green functions for $({}^tL + W_j, D_k), k = 2, 3, \ldots,$ we have $H(x,\cdot) = G_{W_j}(x,\cdot)$. The claim has been proved. By duality, there exists the Green function for $(L + W_j, D)$ with pole at z, which is equal to $H(\cdot,z) = G_{W_j}(\cdot,z)$ on D_m. By Theorem 1.7, G_W and G_{W_j} are comparable (cf. (1.13)). Since m is arbitrary, this together with (2.12) shows (1.8). Then the Lebesgue dominated convergence theorem together with

(1.8) and (2.9) yields (1.7). It remains to prove (1.5), (1.6), and the existence of a homeomorphism Φ. First, we claim that the limit

$$I(x,\xi) \equiv \lim_{y\to\xi} \frac{G_W(x,y)}{G(x_0,y)}, \quad x \in D,$$

exists for any ξ in $\partial_M D_L$. Fix x and choose ℓ such that $x \in D_\ell$. In view of (1.8), the same argument as above shows that for any $\epsilon > 0$ there exists j such that

$$\int_D G_W(x,z)|W_j(z)|G(z,y)dz \le \epsilon G(x_0,y), \quad y \in D_{\ell+1}^c.$$

Since $G(z,y)/G(x_0,y)$ converges to $K(z,\xi)$ as $y \to \xi$ uniformly on any compact set in D, the Fatou lemma yields

$$\int_D G_W(x,z)|W_j(z)|K(z,\xi)dz \le \epsilon.$$

By (1.7),

$$\limsup_{y\to\xi} \frac{G_W(x,y)}{G(x_0,y)} \le K(x,\xi) - \int_D G_W(x,z)W(z)K(z,\xi)dz + 2\epsilon.$$

Similarly,

$$\liminf_{y\to\xi} \frac{G_W(x,y)}{G(x_0,y)} \ge K(x,\xi) - \int_D G_W(x,z)W(z)K(z,\xi)dz - 2\epsilon.$$

Thus

$$\lim_{y\to\xi} \frac{G_W(x,y)}{G(x_0,y)} = K(x,\xi) - \int_D G_W(x,z)W(z)K(z,\xi)dz.$$

By (1.8), the above limit $I(x,\xi)$ is positive. Since

$$\frac{G_W(x,y)}{G_W(x_0,y)} = \frac{G_W(x,y)}{G(x_0,y)} \frac{G(x_0,y)}{G_W(x_0,y)},$$

this means that y converges to a point ξ' in $\partial_M D_{L+W}$. Now we can define Φ by: $\Phi(x) = x$ for $x \in D$, and $\Phi(\xi) = \xi'$ for $\xi \in \partial_M D_L$. Then (1.6) holds and Φ is a continuous map from D_L^* to D_{L+W}^*. Similarly, we can construct a continuous map Ψ from D_{L+W}^* to D_L^*. Since $\Phi \circ \Psi = \Psi \circ \Phi =$ identity in D, the map Φ is a homeomorphism from D_L^* onto D_{L+W}^*. This together with the Martin representation theorem shows that the operator T is a continuous order preserving linear bijection from $H_+(L,D)$ onto $H_+(L+W,D)$. The equality $\Phi(\partial_m D_L) = \partial_m D_{L+W}$ can be proved by using positivity of T. ∎

Proof of Theorem 1.5: First, let us prove (1.9). Since W is G^D_{tL}-semismall at infinity, we can choose, as in the proof of Theorem 1.4, j so large that the inequality

$$\int_D G(x,y)|W_j(y)|G(y,z)dy \le \frac{1}{4}G(x,z), \quad (x,z) \in D \times D_1 \tag{2.13}$$

holds. Then $(L+W_j, D)$ is subcritical, and the inequality

$$\frac{2}{3}G(x,z) \le G_{W_j}(x,z) \le \frac{4}{3}G(x,z), \quad (x,z) \in D \times D_1 \tag{2.14}$$

holds. Let $W_\pm(x) = \max(\pm W(x), 0)$, and $W_{\pm,j} = \chi_j W_\pm$. Since $W_+ - W_{-,j} = W_j + (1-\chi_j)W_+$, we have by Theorem 1.7 and (2.14)

$$C^{-1}G(x,z) \le G_{W_+ - W_{-,j}}(x,z) \le CG(x,z), \quad (x,z) \in D \times D_1, \tag{2.15}$$

for some positive constant C. Since $(L+W, D)$ is critical and $(W_+ - W_{-,j}) - W = (1-\chi_j)W_-$ is a nonnegative function with compact support, we have by Theorem 1.9 of [M2]

$$u(x) = \int_D G_{W_+ - W_{-,j}}(x,y)(1-\chi_j(y))W_-(y)u(y)dy. \tag{2.16}$$

This together with (2.15) implies (1.9). Next, let us prove (1.10). Put

$$v(x) = -\int_D G(x,y)W(y)u(y)dy,$$
$$v_k(x) = -\int_D G_k(x,y)W(y)u(y)dy, \quad k = 1,2,\ldots,$$

where $G_k = G^{D_k}_L$ on D_k^2 and $G_k = 0$ on $D^2 \setminus D_k^2$. By the Harnack inequality and (1.9), there exists a positive constant C such that for any $(x,y) \in D^c_{j+1} \times D$

$$\begin{aligned} G(x,y)|W(y)|u(y) \le & CG(x,y)|W_j(y)|G(y,x_0) \\ & + CG(x,x_0)(1-\chi_j(y))|W(y)|(\sup_{D_j} u). \end{aligned} \tag{2.17}$$

By (2.13), for some positive constant C

$$\int_D G(x,y)|W(y)|u(y)dy \le CG(x,x_0), \quad x \in D^c_{j+1}.$$

By making use of (2.5) with $m = j$,

$$\int_D G(x,y)|W(y)|u(y)dy \le C, \quad x \in D_{j+2}.$$

Thus

$$\int_D G(x,y)|W(y)|u(y)dy \le C\chi_{j+1}(x)G(x,x_0) + C(1-\chi_{j+1}(x)), \quad x \in D. \tag{2.18}$$

Since $G_k \le G$, (2.18) implies that

$$|v_k(x)|, |v(x)| \le C\chi_{j+1}(x)G(x,x_0) + C(1-\chi_{j+1}(x)), \quad x \in D, \tag{2.19}$$

for any $k = 1, 2, \dots$. Since $G_k \to G$ as $k \to \infty$, Lebesgue's dominated convergence theorem shows that

$$\lim_{k\to\infty} v_k(x) = v(x), \quad x \in D. \tag{2.20}$$

Put $w_k = u - v_k$. Since $v_k \in H_0^1(D_k)$, $w_k \in H^1(D_k)$. Furthermore, $Lw_k = -Wu-(-Wu) = 0$ in D_k, and $w_k > 0$ on ∂D_k. Thus, by the maximum principle, $w_k > 0$ on D_k. Then the Harnack inequality together with (2.19) implies that there exists a subsequence of w_k which converges on any compact subset of D to a nonnegative solution w of $Lw = 0$ in D. By (2.20), $w = u - v$. Since w is a nonnegative solution, $w = 0$ or $w(x) > 0$ for any $x \in D$. Suppose that $w(x) > 0$ for any $x \in D$. By (1.9) and (2.19),

$$w(x) \le CG(x,x_0), \quad x \in D_{j+1}^c.$$

This together with the maximum principle shows that $w(x) \le CG(x,x_0)$ for any $x \in D$. Now, put

$$\epsilon = \sup\{t > 0; G(x,x_0) - tw(x) \ge 0 \text{ for any } x \in D\}.$$

Then, $0 < \epsilon < \infty$. Put $h(x) = G(x,x_0) - \epsilon w(x)$. Since $G(x,x_0) \to \infty$ as $x \to x_0$, h is positive in a neighborhood of x_0. Thus $h > 0$ on D. Since $G_k(x,x_0) \to G(x,x_0)$ as $k \to \infty$, there exists a positive constant δ such that $h(x) \ge \delta G(x,x_0)$ for any $x \in D_1^c$. Thus $(\delta/C)w \le h$ on D_1^c; and so $G(x,x_0) - (\epsilon + \delta/C)w(x) \ge 0$ on D. This contradicts the maximality of ϵ. Hence $w = 0$; which is nothing but (1.10). ∎

Proof of Theorem 1.6: We have only to prove (1.11) and (1.12). But they follow from Lemma 2.1 and the proof of Theorem 1.4. ∎

§3. Bounded perturbations

In this section we introduce several notions related to G_L^D-semismallness introduced in Section 1, and show relations among them. As for relevant notions and results, see [BHH] and [Z1,2,3].

Recall that (L,D) is subcritical, G is the Green function for (L,D), and W is a real-valued function in $L_{p,loc}(D)$.

Definition 3.1: We say that W is G_L^D-bounded or G_L^D-semibounded or H_L^D-bounded or H_L^D-semibounded if there exists a positive constant C such that

$$\int_D G(x,y)|W(y)|G(y,z)dy \le CG(x,z), \quad (x,z) \in D^2, \tag{3.1}$$

or

$$\int_D G(x_0,y)|W(y)|G(y,z)dy \le CG(x_0,z), \quad z \in D, \tag{3.2}$$

or

$$\int_D G(x,y)|W(y)|h(y)dy \le Ch(x), \quad x \in D, \quad h \in H_+(L,D), \tag{3.3}$$

or

$$\int_D G(x_0,y)|W(y)|h(y)dy \le Ch(x_0), \quad h \in H_+(L,D), \tag{3.4}$$

respectively.

Definition 3.2: We say that W is $G_L^D H$-integrable if

$$\int_D G(x,y)|W(y)|h(y)dy < \infty, \quad x \in D, \quad h \in H_+(L,D). \tag{3.5}$$

By definition, if W is G_L^D-bounded (or H_L^D-bounded), then it is G_L^D-semibounded (or H_L^D-semibounded); by Propositon 1.3, if W is G_L^D-small at infinity, then it is G_L^D-semismall at infinity. Furthermore, we have the following relations.

PROPOSITION 3.3:

(i) *If W is G_L^D-small (or G_L^D-semismall) at infinity, then W is G_L^D-bounded (or G_L^D-semibounded).*

(ii) *If W is G_L^D-bounded (or G_L^D-semibounded), then W is H_L^D-bounded (or H_L^D-semibounded).*

(iii) *If W is H_L^D-semibounded, then W is $G_L^D H$-integrable.*

Proof: (i) Suppose that W is G_L^D-semismall at infinity. Since $(L+|W|, D)$ is subcritical, we have by (1.7) and (1.8)

$$G(x_0, z) \geq \int_D G_{|W|}(x_0, y)|W(y)|G(y,z)dy \geq C^{-1} \int_D G(x_0, y)|W(y)|G(y,z)dy.$$

Thus W is G_L^D-semibounded. Similarly, if W is G_L^D-small at infinity, then W is G_L^D-bounded.

(ii) Suppose that W is G_L^D-semibounded. For any fundamental sequence $\{z_j\}_{j=1}^{\infty}$ which is a representative of $\xi \in \partial_M D_L$, we have from (3.2)

$$\int_D G(x_0, y)|W(y)|K(y,\xi)dy \leq C.$$

Now, the Martin representation theorem asserts (cf. [CC], [H], [Mae], [Mar], [M2]) that any positive solution $h \in H_+(L, D)$ is represented by an integral on $\partial_M D_L$ with respect to a finite Borel measure μ such that $\mu(\partial_M D_L) = h(x_0)$. Thus we get (3.4). Similarly, if W is G_L^D-bounded, then W is H_L^D-bounded.

(iii) Suppose that W is H_L^D-semibounded. Fix a natural number m. By (2.5), there exists a positive constant C_m such that

$$(3.6) \qquad \int_{D_{m+1}} G(x,y)|W(y)|h(y)dy \leq C_m h(x_0), \quad x \in D_m, \quad h \in H_+(L, D).$$

By the Harnack inequality, there exists a positive constant C'_m such that $G(x,y) \leq C'_m G(x_0, y)$ for any $(x,y) \in D_m \times D^c_{m+1}$, which together with (3.4) yields

$$(3.7) \qquad \int_{D^c_{m+1}} G(x,y)|W(y)|h(y)dy \leq C'_m h(x_0), \quad x \in D_m, \quad h \in H_+(L, D).$$

Combining (3.6) and (3.7), we get (3.5). ∎

Remark 3.4: Suppose that W is G_L^D-bounded with bound $C < 1/2$. Then we see that

$$\frac{1-2C}{1-C} G(x,y) \leq G_W(x,y) \leq \frac{1}{1-C} G(x,y), \quad x, y \in D.$$

By Theorem 2.3 of [P1], this implies that there exists a homeomorphism Φ from the minimal Martin boundary $\partial_m D_L$ onto $\partial_m D_{L+W}$ such that (1.11) holds with $\partial_M D_L$ replaced by $\partial_m D_L$. Therefore, the minimal Martin boundary is stable under G_L^D-bounded perturbations with bounds smaller than $1/2$. In particular, if $\partial_M D_L$ consists of one point, then $\partial_M D_{L+W}$ also does.

Another notion related H_L^D-boundedness is intrinsic ultracontractivity to be defined below. Suppose that L is formally self-adjoint, i.e., $b_j \equiv 0$ for any j. Since (L, D) is subcritical, the quadratic form on $C_0^\infty(D)$ associated with L is nonnegative; and there exists a nonnegative self-adjoint operator L_D on $L_2(D)$ corresponding to L (cf. [D]). Let $\Gamma(x, y, t)$ be a minimal fundamental solution for $\partial_t + L$ on $D \times (0, \infty)$; it is an integral kernel of the semigroup e^{-tL_D} (see, for example, [M5]). Denote by λ_0 the infimum of the spectrum of L_D. Following [DS], e^{-tL_D} is said to be intrinsically ultracontractive (in short, IU) when the following conditions are satisfied: (i) λ_0 is a positive eigenvalue of L_D with strictly positive eigenfunction ϕ_0 normalized by $\|\phi_0\|_2 = 1$; and (ii) for each $t > 0$, there exists a positive constant C_t such that

$$C_t^{-1}\phi_0(x)\phi_0(y) \le \Gamma(x, y, t) \le C_t\phi_0(x)\phi_0(y), \quad (x, y) \in D^2. \tag{3.8}$$

PROPOSITION 3.5: *If e^{-tL_D} is IU, then the constant function 1 is G_L^D-bounded.*

Proof: Since

$$\int_D \Gamma(y, z, s)\phi_0(y)dy = e^{-\lambda_0 s}\phi_0(z),$$

we have by (3.8) with $t = 1$

$$C_1^{-1}e^{-\lambda_0 s}\phi_0(x)\phi_0(z) \le \Gamma(x, z, s+1) \le C_1 e^{-\lambda_0 s}\phi_0(x)\phi_0(z). \tag{3.9}$$

By the maximum principle (cf. [Ar]),

$$\int_D \Gamma(x, y, t)G(y, w)dy \le G(x, w), \quad (x, w) \in D^2, \quad t > 0. \tag{3.10}$$

By (3.8) and (3.10),

$$\phi_0(x)\int_D \phi_0(y)G(y, w)dy \le C_1 G(x, w). \tag{3.11}$$

By (3.9) and (3.11),

$$\int_D \Gamma(x, z, s)G(z, w)dz \le (C_1^2 e^{\lambda_0})e^{-\lambda_0 s}G(x, w), \quad s > 1, \ (x, w) \in D^2. \tag{3.12}$$

Finally, by (3.10) and (3.12),

$$\int_D G(x,z)G(z,w)dz = \int_0^\infty ds \int_D \Gamma(x,z,s)G(z,w)dz$$
$$\leq [1 + C_1^2 e^{\lambda_0} \int_1^\infty e^{-\lambda_0 s} ds] G(x,w).$$

This shows (3.1) with $W = 1$. ∎

Remark: Propositions 3.3 and 3.5 show that if e^{-tL_D} is IU, then 1 is H_L^D-bounded. This, however, is not really new (cf. the sentence below the proof of Proposition 1 of [B3]).

Here we should mention a historical remark.

Remark: The estimate (3.3) with $W = 1$ is called a (conditioned) life time estimate or Cranston–McConnell estimate, since they [CM] first established such an estimate in connection with the (conditioned) expectation of the exit time τ_D of the Brownian motion. Recently, life time estimates and intrinsic ultracontractivity have been investigated extensively by many probabilists and analysts (cf. [AM], [B1,2,3], [BC], [BD1,2], [BØ], [Ci], [CM], [D], [DS], [F], [M4,5,6], and references therein). Bañuelos and Davis [BD1,2] gave, among others, examples which are not IU, but for which life time estimates hold. On the other hand, Murata [M4,5,6] observed that for an unbounded domain D, intrinsic ultracontractivity or the estimate (3.5) with $W = 1$ implies existence of a positive solution of a parabolic equation with zero initial and boundary value; and gave sharp criteria for the uniqueness of the positive Cauchy problem.

We say that e^{-tL_D} is semi-IU when the conditions (i) and (ii) hold with (3.8) replaced by

$$C_t^{-1}\phi_0(y) \leq \Gamma(x_0, y, t) \leq C_t \phi_0(y), \quad y \in D. \tag{3.8'}$$

The following proposition can be shown in the same way as in the proof of Theorem 5.1 of [M5].

PROPOSITION 3.6: *The semigroup e^{-tL_D} is semi-IU if and only if 1 is G_L^D-semibounded.*

§4. Sufficient conditions for a perturbation to be G_L^D-semismall at infinity

In this section we give two conditions sufficient for W to be G_L^D-semismall at infinity. Their applications will be given in the next section.

We first give a sufficient condition coming from the basic estimate developed in connection with life time estimates (cf. [Ai], [AM], [B3]).

THEOREM 4.1:

(i) *Let $x \in D$ and $\eta > 1$. Let h be a positive continuous solution of the equation*

$$ {}^tLh = -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j h) + \sum_{j=1}^{n} \partial_j(b_j h) + Vh = 0 \quad \text{in } D. $$

For any integer k, put

$$ E_k^x = \{y \in D; \eta^{k-1} < h(x)G(x,y)h(y)^{-1} < \eta^{k+2}\}, $$
$$ F_k^x = \{y \in D; \eta^{k} \le h(x)G(x,y)h(y)^{-1} \le \eta^{k+1}\}. $$

Then

$$ \sup_{z\in D} \frac{1}{G(x,z)} \int_D G(x,y)f(y)G(y,z)dy \le \frac{\eta^3}{(\eta-1)^2} \sum_{k=-\infty}^{\infty} \sup_{w\in F_k^x} \frac{1}{h(w)} \int_{F_k^x} G_L^{E_k^x}(y,w)h(y)f(y)dy \tag{4.1} $$

for any nonnegative measurable function f on D.

(ii) *Let $x \in D$ and $\eta > 1$. Suppose that either there exists a positive continuous solution h of the equation ${}^tLh = 0$ in D such that*

$$ \eta^{-1} \le h(y)/h(x) \le \eta, \quad y \in D, \tag{4.2} $$

or

$$ \sum_{j=1}^{n} \partial_j b_j + V \ge 0 \quad \text{in } D \tag{4.3} $$

in the distribution sense. For any integer k, put

$$ A_k^x = \{y \in D; \eta^{k-2} < G(x,y) < \eta^{k+3}\}, $$
$$ B_k^x = \{y \in D; \eta^{k-1} \le G(x,y) \le \eta^{k+2}\}. $$

Then

$$\sup_{z\in D}\frac{1}{G(x,z)}\int_D G(x,y)f(y)G(y,z)dy \le \frac{\eta^5}{(\eta-1)^2}\sum_{k=-\infty}^{\infty}\sup_{w\in B_k^x}\int_{B_k^x} G_L^{A_k^x}(y,w)f(y)dy \tag{4.4}$$

for any nonnegative measurable function f on D.

Proof: (i) Recall that $\{D_j\}_{j=1}^{\infty}$ is the exhaustion of D given in Section 1. Put $G_j = G_L^{D_j}$, $D_jE_k^x = D_j\cap E_k^x$, $D_jF_k^x = D_j\cap F_k^x$, and $G_{j,k} = G_L^{D_jE_k^x}$. Set

$$^tL_h = -h^{-2}\sum_{i,j=1}^{n}\partial_i(h^2a_{ij}\partial_j) + \sum_{j=1}^{n} b_j\partial_j. \tag{4.5}$$

Then we have

(4.6)
$$G_{^tL_h}^{D_jE_k^x}(w,y) = h(w)^{-1}G_{j,k}(y,w)h(y),$$
$$G_{^tL_h}^{D_j}(y,x) = h(y)^{-1}G_j(x,y)h(x),\quad G_{^tL_h}^{D}(y,x) = h(y)^{-1}G(x,y)h(x).$$

Since 1 is a positive supersolution of the equation $^tL_hv=0$ in D, the usual weak maximum principle for ^tL_h holds in any open set $\Omega\Subset D$; that is, if $v\in H^1_{loc}(\Omega)$ satisfies, for some positive constant M,

$$^tL_hv\le 0 \text{ in } \Omega,\quad v\le M \text{ on } \partial\Omega,$$

then $v\le M$ on Ω. Thus we can apply the same argument as in the proof of Theorem 3 (the generalized basic estimate) of [AM] (see the formulas (7) and (10) therein) to show that for any j,k

$$\sup_{z\in D_j}\frac{1}{u(z)}\int_{D_jF_k^x} G_{^tL_h}^{D_j}(z,y)f(y)u(y)dy \le\frac{\eta^2}{(\eta-1)^2}\sup_{w\in D_jF_k^x}\frac{1}{u(w)}\int_{D_jF_k^x} G_{^tL_h}^{D_jE_k^x}(w,y)f(y)u(y)dy, \tag{4.7}$$

where

$$u(y) = h(y)^{-1}G(x,y)h(x).$$

Since $u(y)/u(w) \le \eta$ for any $y, w \in F_k^x$ and $\bigcup_k F_k^x = D \setminus \{x\}$, we have by (4.6) and (4.7)

$$\int_{D_j} G_j(y,z) f(y) G(x,y) dy$$
$$\le \frac{\eta^3}{(\eta-1)^2} G(x,z) \sum_{k=-\infty}^{\infty} \sup_{w \in D_j F_k^x} \int_{D_j F_k^x} h(w)^{-1} G_{j,k}(y,w) h(y) f(y) dy$$

for any $z \in D_j$. Thus, letting $j \to \infty$, we get (4.1).

(ii) Assume the existence of a positive solution h satisfying (4.2). Then $E_k^x \subset A_k^x$, $F_k^x \subset B_k^x$, and $h(y)/h(w) \le \eta^2$ for any $y, w \in D$. Thus (4.4) follows from (4.1). Finally, assume (4.3). Then 1 is a positive supersolution of the equation ${}^t L_h v = 0$ in D. Thus, as in (i), we get

$$\sup_{z \in D} \frac{1}{G(x,z)} \int_D G(x,y) f(y) G(y,z) dy \tag{4.8}$$
$$\le \frac{\eta^3}{(\eta-1)^2} \sum_{k=-\infty}^{\infty} \sup_{w \in b_k^x} \int_{b_k^x} G_L^{a_k^x}(y,w) f(y) dy,$$

where

$$a_k^x = \{y \in D; \eta^{k-1} < G(x,y) < \eta^{k+2}\} \quad \text{and} \quad b_k^x = \{y \in D; \eta^k \le G(x,y) \le \eta^{k+1}\}.$$

Obviously, (4.4) follows from (4.8). ∎

Denote by $m_h(f,x)$ and $M(f,x)$ the right hand sides of (4.1) and (4.4), respectively. In the proof of Theorem 4.1, we have shown that if (4.2) is satisfied, then

$$m_h(f,x) \le M(f,x); \tag{4.9}$$

and that if (4.3) is satisfied, then (4.9) also holds with $h = 1$. Put $W_j = W \chi_{D_j^c}$. Then we have the following theorem.

THEOREM 4.2: *If*

$$\lim_{j\to\infty} m_h(|W_j|, x_0) = 0 \quad (\textit{or } \lim_{j\to\infty} \sup_{x \in D_j^c} m_h(|W_j|, x) = 0),$$

then W is G_L^D-semismall (or G_L^D-small) at infinity. Furthermore, under the condition (4.2) with $x = x_0$ or (4.3), if

$$\lim_{j\to\infty} \sup_{y \in D_j^c} G(x_0, y) = 0 \quad \textit{and} \quad M(|W|, x_0) < \infty,$$

then W is G_L^D-semismall at infinity.

Proof: We have only to prove the last statement. For $j = 1, 2, \ldots$, put

$$k(j) = \min\{k \in \mathbb{Z}; \sup_{y \in D_j^c} G(x_0, y) \le \eta^{k+2}\}.$$

Then $k(j) \to -\infty$ as $j \to \infty$, for $\sup\{G(x_0, y); y \in D_j^c\} \to 0$ as $j \to \infty$. Furthermore,

$$\{k \in \mathbb{Z}; B_k^{x_0} \cap D_j^c \neq \emptyset\} \subset \{k \in \mathbb{Z}; k \le k(j)\}.$$

This together with (4.4) implies that

$$\sup_{z \in D_j^c} \frac{1}{G(x_0, z)} \int_{D_j^c} G(x_0, y)|W(y)|G(y, z)dy \le M(|W_j|, x_0)$$
$$\le \frac{\eta^5}{(\eta - 1)^2} \sum_{k=-\infty}^{k(j)} \sup_{w \in B_k^{x_0}} \int_{B_k^{x_0} \cap D_j^c} G_L^{A_k^{x_0}}(y, w)|W(y)|dy.$$

Since $M(|W|, x_0) < \infty$ and $\lim_{j \to \infty} k(j) = -\infty$, the above inequality yields (1.3). ∎

We next give a sufficient condition which is suggested by the 3G theorem developed in connection with the gauge theorem (cf. [BHH], [CFZ], [HZ], [Z1,2,3], and references therein) in a rather general form, expecting further applications.

Definition 4.3: Let F be a nonnegative measurable function on D^2. If there exists a positive constant C such that

(4.10) $$G(x, y)G(y, z) \le CG(x, z)\{F(x, y) + F(y, z)\}, \quad x, y, z \in D,$$

then we call (4.10) a 3G inequality with F.

For any nonnegative measurable function f on D, put

(4.11) $$N_F(f, x, z) = \int_D \{F(x, y) + F(y, z)\}f(y)dy.$$

Clearly, we have

THEOREM 4.4: *Suppose that there holds a 3G inequality with F. If*

(4.12) $$\lim_{j \to \infty} \sup_{x, z \in D_j^c} N_F(|W_j|, x, z) = 0 \quad \left(\text{or } \lim_{j \to \infty} \sup_{z \in D_j^c} N_F(|W_j|, x_0, z) = 0\right),$$

then W is G_L^D-small (or G_L^D-semismall) at infinity.

Here, we should mention a weak form of (4.10) with $F = 1$ (cf. [P1, Lemma 2.11]): For any $\delta > 0$ with $B(x_0, \delta) \equiv \{x \in \mathbb{R}^n; |x - x_0| < \delta\} \subset\subset D$, there exists $C > 0$ such that

$$G(x, x_0)G(x_0, z) \leq CG(x, z), \quad x, z \in D \setminus B(x_0, \delta). \tag{4.13}$$

We conclude this section with a remark on the regularity condition on the coefficients, which is concerned with all results in this paper.

Remark 4.5: Even if the condition that $V, W \in L_{p,loc}(D)$ is replaced by a less stringent one, that V and W belong to the local Kato class $K_n^{loc}(D)$, all results given in the preceding sections and those to be given in the next section are still valid, because Green functions in this case are still comparable with the standard one (cf. (2.5) in Section 2). For results concerning the Kato class, see [K], [B2], [BHH], [CFZ], and [Z1,2,3]. But we do not know whether the condition that $b_j \in L_{2p,loc}(D)$ can be replaced by a less stringent one, that $|b_j|^2 \in K_n^{loc}(D)$.

§5. Examples

In this section we give several examples concerning G_L^D-semismallness.

Throughout this section we assume that L is uniformly elliptic, i.e., $(a_{ij}(x))_{i,j=1}^n$ is a matrix-valued measurable function satisfying

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x, \xi \in \mathbb{R}^n. \tag{5.1}$$

In what follows we denote by $|B|$ the Lebesgue measure of a Borel set B in $\mathbb{R}^n$.

5.1 Throughout this subsection we further assume that D is a bounded domain, $b_j \in L_{2p}(D)$ and $V \in L_p(D)$. In this case, where D is bounded, the term (G_L^D-semismall) *at infinity* may be read as (G_L^D-semismall) *at the boundary* ∂D.

THEOREM 5.1: *Suppose that there exists a positive solution $h \in H_+({}^tL, D)$ such that $h(x_0) = 1$ and $\{h(x); x \in D\} \subset (\eta^{-1}, \eta)$ for some $\eta > 1$. Put*

$$A_k = \{y \in D; \eta^{k-2} < G(x_0, y) < \eta^{k+3}\}, \tag{5.2}$$

$$I_j = \{k \in \mathbb{Z}; A_k \cap D_j^c \neq \emptyset\}, \quad j = 1, 2, \ldots. \tag{5.3}$$

If

(5.4)
$$\lim_{j\to\infty}\sum_{k\in I_j}\|W\|_{L_p(A_k\cap D_j^c)}|A_k|^{2/n-1/p}=0,$$

then W is G_L^D-semismall at infinity.

Proof: Put $W_j=|W|\chi_{D_j^c}$. By (4.4),

(5.5)
$$\sup_{z\in D}\frac{1}{G(x_0,z)}\int_D G(x_0,y)W_j(y)G(y,z)dy\le C\sum_k\sup_{w\in A_k}\int_{A_k}G_L^{A_k}(y,w)W_j(y)dy.$$

Let us show that there exists a positive constant C such that for any j and $k\in I_j$

(5.6)
$$\sup_{w\in A_k}\int_{A_k}G_L^{A_k}(y,w)W_j(y)dy\le C\|W\|_{L_p(A_k\cap D_j^c)}|A_k|^{2/n-1/p}.$$

Choose q such that $n/2<q<\min(n,p)$, and put $r=(1/q-1/n)^{-1}$. First, consider the Dirichlet problem

$${}^tLu=\sum_{i=1}^n\partial_if_i\quad\text{in }\Omega,\quad u\in H_0^1(\Omega),$$

where Ω is an open subset of D and $f_i\in L_r(\Omega), i=1,\dots,n$. Clearly, $n<r<\infty$. By Théorème 4.2 and Remark 4.3 in [S], there exists K and N independent of Ω such that

$$\sup_{x\in\Omega}|u(x)|\le K\sum_{i=1}^n\|f_i\|_{L_r(\Omega)}|\Omega|^{1/n-1/r}+N\|u\|_{L_2}.$$

Next, we claim that there exists a positive constant C independent of Ω such that

$$\|u\|_{L_2(\Omega)}\le C\sum_{i=1}^n\|f_i\|_{L_2(\Omega)}.$$

In order to prove this claim, we prepare some notations. Denote by ${}^t\tilde L_\Omega$ the Dirichlet realization of ${}^tL$ in $L_2(\Omega)$. That is, ${}^t\tilde L_\Omega u={}^tLu$ for u in the domain of ${}^t\tilde L_\Omega$, which is equal to $\{u\in H_0^1(\Omega);{}^tLu\in L_2(\Omega)\}$. Let $\sigma({}^t\tilde L_\Omega)$ be the spectrum of ${}^t\tilde L_\Omega$, and put

$$\Gamma({}^tL,\Omega)=\inf\{\text{Re }z;z\in\sigma({}^t\tilde L_\Omega)\}.$$

Since D is bounded, $b_j\in L_{2p}(D), V\in L_p(D)$, and there exists a solution h satisfying $\eta^{-1}<h(x)<\eta$, Theorems 1.4 and 1.5 of [M2] show that

$$\Gamma({}^tL,\Omega)\ge\Gamma({}^tL,D)>0.$$

Then, by Théorème 3.4 in [S], the operator $({}^t\tilde{L}_\Omega)^{-1}$ is extended to a bounded operator from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$; furthermore,

$$({}^t\tilde{L}_\Omega)^{-1} = \sum_{k=0}^{\infty} \lambda^k [({}^t\tilde{L}_\Omega + \lambda)^{-1}]^{k+1},$$

where λ is a positive number such that the form $(({}^tL + \lambda)\varphi, \varphi)$ on $H_0^1(D)$ is coercive. This proves the claim. Since

$$\|f_i\|_{L_2(\Omega)} \le \|f_i\|_{L_r(\Omega)} |\Omega|^{1/2-1/r}$$

and $|\Omega|^{1/2-1/n} \le |D|^{1/2-1/n}$, we thus get

$$\sup_{x\in\Omega} |u(x)| \le (K + N|D|^{1/2-1/n}) \sum_{i=1}^{n} \|f_i\|_{L_r(\Omega)} |\Omega|^{1/n-1/r}.$$

Finally, consider the Dirichlet problem

$${}^tLu = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

where $f \in L_q(\Omega)$. Let g be a solution of the Dirichlet problem

$$-\Delta g = \tilde{f} \quad \text{in } B, \quad g \in H_0^1(B),$$

where B is an open ball including $\overline{D}$, $\tilde{f} = f$ in Ω and $\tilde{f} = 0$ on $B \setminus \Omega$. Put $f_i = -\partial_i g, i = 1, \ldots, n$. Then,

$$f = \sum_{i=1}^{n} \partial_i f_i, \quad \sum_{i=1}^{n} \|f_i\|_{L_r(\Omega)} \le C\|f\|_{L_q(\Omega)},$$

where C is a positive constant. Thus, we have

$$\sup_{x\in\Omega} |u(x)| \le C\|f\|_{L_q(\Omega)} |\Omega|^{2/n-1/q} \le C\|f\|_{L_p(\Omega)} |\Omega|^{2/n-1/p}$$

with another positive constant C. This completes the proof of (5.6), since

$$u(x) = \int_{A_k} G_L^{A_k}(y, x) W_j(y) dy$$

is a solution of the Dirichlet problem

$${}^tLu = W_j \quad \text{in } A_k, \quad u \in H_0^1(A_k).$$

Hence, by (5.5) and (5.6),

(5.7)

$$\sup_{z\in D} \frac{1}{G(x_0,z)} \int_{D_j^c} G(x_0,y)|W(y)|G(y,z)dy \le C \sum_{k\in I_j} \|W\|_{L_p(A_k\cap D_j^c)} |A_k|^{2/n-1/p}.$$

This proves the theorem. ∎

Now, suppose that ∂D is regular with respect to the Dirichlet problem for (L, D) in the sense that $G(x_0, y) \to 0$ as $y \to \partial D$. Then we see that there exists such a positive solution h as in Theorem 5.1, and that $\sup I_j \to -\infty$ as $j \to \infty$. Hence we have

THEOREM 5.2: *Suppose that ∂D is regular with respect to the Dirichlet problem. If*

$$\sum_{k=-\infty}^{0} \|W\|_{L_p(A_k)} |A_k|^{2/n-1/p} < \infty, \tag{5.8}$$

then W is G_L^D-semismall at infinity.

Example 5.3: Let $n = 2$. If ∂D is regular with respect to the Dirichlet problem, then 1 is G_L^D-semismall at infinity. In fact, the left hand side of (5.8) is estimated from above by $\sum_{k=-\infty}^{0} |A_k| \le 5|D|$.

THEOREM 5.4.1: *Suppose that D is a bounded Lipschitz domain. For any $r > 0$, put $D^r = \{x \in D; d(x) \equiv \operatorname{dist}(x, \partial D) > r\}$. Then there exists $\alpha > 0$ such that if W satisfies*

$$\|W\|_{L_p(D^r)} \le Cr^{-\alpha}, \quad r > 0, \tag{5.9}$$

for some $C > 0$, then W is G_L^D-semismall at infinity.

Proof: We see that there exist positive constants β, γ, C such that

$$C^{-1}d(y)^{\beta} \le G(x_0, y) \le Cd(y)^{\gamma}$$

for any y with $d(y) << 1$. (For the first inequality, see [B2] and [F, Proposition 1]; and for the second, see [S].) Thus

$$(C^{-1}\eta^{k-2})^{1/\gamma} < d(y) < (C\eta^{k+3})^{1/\beta}, \quad y \in A_k.$$

Since $|D \setminus D^r| < Cr$ for $r << 1$, there exists $\alpha > 0$ such that (5.9) implies (5.8). Hence the theorem follows from Theorem 5.2. ∎

Example 5.4.2: Suppose that $b_j = 0$, $V = 0$, and D is a bounded Lipschitz domain. First, let $n \geq 3$. Then, Theorem 3.1 of [CFZ] asserts that the 3G inequality (4.10) with $F(x,y) = |x-y|^{2-n}$ holds. Thus, if

$$\lim_{j\to\infty} \sup_{x\in D_j^c} \int_{D_j^c} |x-y|^{2-n}|W(y)|dy = 0, \tag{5.10}$$

then W is G_L^D-small at infinity. Second, let $n = 2$. Then it follows from the proof of the above Theorem 3.1 that

$$G(x,y)G(y,z) \leq CG(x,z)\left\{\log\frac{R}{|x-y|} + \log\frac{R}{|y-z|}\right\}, \tag{5.11}$$

where $R = \sup\{3|x-y|; (x,y) \in D^2\}$. Thus, if

$$\lim_{j\to\infty} \sup_{x\in D_j^c} \int_{D_j^c} \left(\log\frac{R}{|x-y|}\right)|W(y)|dy = 0, \tag{5.12}$$

then W is G_L^D-small at infinity.

5.2 In this subsection we assume that $L = -\sum_{i,j=1}^n \partial_i a_{ij}(x)\partial_j$ with the coefficients satisfying (5.1).

Example 5.5: Suppose that $D = \mathbb{R}^n$, $n \geq 3$. Then it is well-known that $G(x,y)$ is comparable with $|x-y|^{2-n}$. Thus we easily see that (4.10) holds with $F(x,y) = |x-y|^{2-n}$ (see also [BHH, Lemma 7.3]). Hence, if

$$\lim_{j\to\infty} \sup_{|x|\geq j} \int_{|y|\geq j} |x-y|^{2-n}|W(y)|dy = 0, \tag{5.13}$$

then W is G_L^D-small at infinity (see also [Z1,2,3]).

Example 5.6: For some $\alpha < -1$, put

$$D = \{x = (x_1, x') \in \mathbb{R}^n; x_1 > 1, |x'| < x_1^\alpha\}.$$

Then 1 is G_L^D-semismall at infinity. This can be proved in a way similar to the proof of Theorem 5.2 (see also the proof of Theorem 1.1 in [M6]).

Example 5.7: Let $n = 2$, $a_{jj} = \lambda_j(x_j)$, $j = 1, 2$, and $a_{12} = a_{21} = 0$. Suppose that $D \subset \{x \in \mathbb{R}^2; x_1 > 0, x_2 > 0\}$, and ∂D is regular with respect to the Dirichlet problem. Assume that

$$|W(x)| \leq C(x_1^\alpha + x_2^\beta)$$

for some real numbers α and β. Then the function

$$\int_0^{x_j} \frac{dt}{\lambda_j(t)}$$

is a positive solution of $Lu = 0$ in D, and comparable with x_j. Thus, it follows from the proof of Theorems 1 and 4 of [AM] (which is based upon the basic estimate like (4.4)) that if

$$\int_D (x_1^\alpha + x_2^\beta)dx < \infty,$$

then W is G_L^D-small at infinity.

5.3 In this subsection we assume that $D = \mathbb{R}^n$.

Example 5.8: Let $L = -\Delta + |x|^\beta$ for some $\beta > -2$. Suppose that

$$(1 + |x|)^{1-\beta/2+\epsilon-n/q} W(x) \in L_q(\mathbb{R}^n)$$

for some $q \geq n$ and $\epsilon > 0$. Then the proof of Theorem 5.8 of [M1] shows that if $-2 < \beta \leq 0$ (or $\beta > 0$), then W is G_L^D-small (or G_L^D-semismall) at infinity. (As for the case $\beta = 0$, see also [HZ].)

THEOREM 5.9: *Suppose that*

$$L = -\sum_{i,j=1}^n \partial_i a_{ij}(x)\partial_j + V(x),$$

where $\lambda \leq V \leq \mu$ for some positive constants λ and μ. Let $E(y,w) = |y-w|^{2-n}$ for $n \geq 3$, and $E(y,w) = \log(2/|y-w|)$ for $n = 2$. Then there exists a natural number N such that if

$$\sum_{k=N}^{\infty} \sup_{\frac{k}{N} < |w| < Nk} \int_{|y-w|<1} E(y,w)|W(y)|dy < \infty, \tag{5.14}$$

then W is G_L^D-semismall at infinity.

Proof: Put $L_0 = -\sum_{i,j} \partial_i a_{ij}(x)\partial_j$, $G_{L_0+\lambda} = G^{\mathbb{R}^n}_{L_0+\lambda}$, and let $\Gamma(x,y,t)$ be the fundamental solution for the parabolic equation $(\partial_t - L_0)u = 0$ on $\mathbb{R}^n \times (0,\infty)$. Then there exist positive numbers α, β, C such that

$$C^{-1}t^{-n/2}\exp\left(-\frac{\alpha|x-y|^2}{t}\right) \leq \Gamma(x,y,t) \leq Ct^{-n/2}\exp\left(-\frac{\beta|x-y|^2}{t}\right)$$

(cf. [Ar]). Since $V \geq \lambda$, this together with the maximum principle yields

$$G(x,y) \leq G_{L_0+\lambda}(x,y) = \int_0^\infty \Gamma(x,y,t)e^{-\lambda t}dt$$
$$\leq C\int_0^\infty t^{-n/2}\exp\left(-\frac{\beta|x-y|^2}{t} - \lambda t\right)dt \leq C_1 g(x,y;4\beta\lambda),$$

where $g(x,y;4\beta\lambda)$ is the Green function for $(-\Delta + 4\beta\lambda, \mathbb{R}^n)$ and C_1 is a positive constant. Similarly,

$$C_1^{-1}g(x,y;4\alpha\mu) \leq G(x,y).$$

Thus we have:

(i) for $|x-y|<1$,

$$C^{-1}E(x,y) \leq G(x,y) \leq CE(x,y); \tag{5.15}$$

(ii) for $|x-y| \geq 1$,

$$C^{-1}e^{-\gamma|x-y|} \leq G(x,y) \leq Ce^{-\delta|x-y|}, \tag{5.16}$$

where γ and δ are positive constants.

Let us apply Theorem 4.1(ii). For any $k>5$, put

$$Z_k = A^0_{-k} \equiv \{y \in D; \eta^{-k-2} < G(0,y) < \eta^{-k+3}\}.$$

In view of (5.16), we can and will choose $\eta > 1$ and $N > 1$ such that

$$Z_k \subset Y_k \equiv \{y \in \mathbb{R}^n; k/N < |y| < Nk\}$$

for any $k>5$. This together with (4.4) yields

(5.17)
$$\sup_{z\in D}\frac{1}{G(0,z)}\int_D G(0,y)W_j(y)G(y,z)dy \leq C\sum_{k>N}\sup_{w\in Y_k}\int_{Y_k}G(y,w)W_j(y)dy$$

for any sufficiently large j, where $W_j = |W|\chi_{D_j^c}$. By (5.15) and (5.16), the right hand side of (5.17) is estimated from above by

$$C\left[\sup_{w\in Y_k}\int_{|y-w|<1}E(y,w)W_j(y)dy\right] \times \left[1 + \sum_{l=1}^{2Nk} e^{-\delta l}l^{n-1}\right].$$

Thus

(5.18)
$$\sup_{z\in D}\frac{1}{G(0,z)}\int_D G(0,y)W_j(y)G(y,z)dy \leq C\sum_{k>N}\sup_{w\in Y_k}\int_{|y-w|<1}E(y,w)W_j(y)dy$$

with another constant C. Since $\inf\{k; D_j^c \cap Y_k\} \to \infty$ as $j\to\infty$, (5.14) and (5.18) complete the proof of the theorem. ∎

Example 5.10: Let L be as above. Suppose that the coefficients a_{ij} and V are periodic functions, W satisfies (5.14), and $(L+W, D)$ is subcritical. Then, by Theorem 1.4 in Section 1 and results of [Ag] (see also [LP]), the minimal Martin boundary $\partial_m D_{L+W}$ for $(L+W, \mathbb{R}^n)$ is homeomorphic to the boundary of a non-empty strictly convex bounded open subset of $\mathbb{R}^n$.

THEOREM 5.11: *Let $L = -\Delta + V$ with $\lambda|x|^\beta \le V \le \mu|x|^\beta$ for some positive numbers β, λ, and μ. Then there exists a natural number N such that if*

$$\sum_{k=N}^{\infty} \sup_{\frac{k}{N}<|w|^{\beta/2+1}<Nk} \int_{|y-w|<|w|^{-\beta/2}} E(y,w)|W(y)|dy < \infty, \tag{5.19}$$

then W is G_L^D-semismall at infinity.

Proof: For $a > 0$, let H_a be the Green function for $(-\Delta + a|x|^\beta, \mathbb{R}^n)$. Then $H_\lambda \le G \le H_\mu$. By Theorems 4.10 and 3.12 of [M1], we have the properties of H_a which are necessary for carrying out the same argument as in the proof of Theorem 5.9; for example,

$$H_a(0,y) = C|y|^{-(n-1)/2-\beta/4}\exp[-a^{1/2}(\beta/2+1)^{-1}|y|^{\beta/2+1}][1+o(1)] \text{ as } |y| \to \infty.$$

Hence the theorem can be shown as Theroem 5.9. We omit the details. ∎

ACKNOWLEDGEMENT: The author cordially thanks Y. Pinchover and K. Kurata for valuable conversations, and is greatly indebted to the referee and H. Aikawa for their criticisms which improved the original version of this paper.

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